ON THE LUNAR INEQUALITIES PRODUCED BY THE MOTION OF THE ECLIPTIC.

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THIS subject has been treated by Hansen,* and more recently by Sir G. B. Airy and Prof. J. C. Adams.† Hansen's discussion is accommodated to the peculiar system of co-ordinates he employs, and the two later writers do not consider the inequalities in longitude. Hence an investigation, giving the inequalities of the latitude and longitude, at first, in the literal form, may be of value. The procedures employed are very similar to those of Pontécoulant, and are doubtless not as direct as might be imagined. The paper was written as long ago as 1867.

I.

Expressed in the ordinary notation, when the co-ordinates are referred to fixed planes, the differential equations of motion are

$$\frac{d^2X}{dt^2}+\frac{\mu}{r^3}X=\frac{dR}{dx},$$

$$\frac{d^2Y}{dt^2}+\frac{\mu}{r^3}Y=\frac{dR}{dy},$$

$$\frac{d^2Z}{dt^2}+\frac{\mu}{r^3}Z=\frac{dR}{dz}.$$

Since the directions of the axes are arbitrary, let the axis of X be directed toward the ascending node of the moving ecliptic on the ecliptic of 1850; and let the axis of Z be perpendicular to the latter plane. Taking now another system of co-ordinates, x, y, and z, such that the axis of x has the same direction as that of X, but the axis of z is perpendicular to the moving ecliptic, let $\pi\,(t-1850)$ be the inclination of the moving ecliptic to that of 1850; then, neglecting quantities of the order of π^2, these equations exist,

$$X=x,$$

$$Y=y-\pi\,(t-1850)\,z,$$

$$Z=z+\pi\,(t-1850)\,y.$$

The differential equations of motion, expressed in terms of the second system of co-ordinates, are

$$\frac{d^2x}{dt^2}+\frac{\mu}{r^3}x=\frac{dR}{dx},$$

$$\frac{d^2y}{dt^2}+\frac{\mu}{r^3}y=\frac{dR}{dy}+2\pi\frac{dz}{dt},$$

$$\frac{d^2z}{dt^2}+\frac{\mu}{r^3}z=\frac{dR}{dz}-2\pi\frac{dy}{dt}.$$

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* *Darlegung* etc., Arts. 175–178.

† *Monthly Notices*, Vol. XLI., pp. 264, 375, and 385.

Denoting the true longitude of the moon by λ, from these may be derived the two

$$\frac{d^2r^2}{2\,dt^2} - \frac{\mu}{r} + \frac{\mu}{a} = 2\int d'R + r\frac{dR}{dr} + 2\pi\frac{ydz - zdy}{dt},$$

$$\frac{d.(r^2 - z^2)\dfrac{d\lambda}{dt}}{dt} = \frac{dR}{d\lambda} + 2\pi\frac{xdz}{dt}.$$

In this discussion all terms involving the solar eccentricity and parallax will be neglected. Let ζ denote the moon's mean angular distance from a point 90° behind the ascending node of the moving ecliptic on the ecliptic of 1850, or $\zeta = \varepsilon + nt - \Pi + 90°$. For simplicity, the semi-axis major of the lunar orbit will be made equal to unity, and, as usual in the lunar theory, m will be written for $\frac{n'}{n}$. Also let φ and τ denote respectively the true and mean angular distance of the moon from the sun.

With these restrictions and notation

$$2\int d'R + r\frac{dR}{dr} = 4R + 2m\int\frac{dR}{d\lambda}d\zeta,$$

$$R = \frac{m^2}{4}[3\,(r^2 - z^2)\cos 2\varphi + r^2 - 3z^2],$$

$$\frac{dR}{d\lambda} = -\frac{3}{2}m^2\,(r^2 - z^2)\sin 2\varphi.$$

If the symbol δ prefixed to any quantity denote that part of it, in its development in series, which is multiplied by the first power of π, the equations for determining δr, $\delta\lambda$, and δz are

$$\frac{d^2 r\delta r}{d\zeta^2} + \frac{\mu r\delta r}{n^2r^3} = 4\delta R + 2m\int\delta.\frac{dR}{d\lambda}d\zeta + 2\frac{\pi}{n}\frac{ydz - zdy}{d\zeta},$$

$$\frac{r^2 d.\delta\lambda}{d\zeta} + 2\frac{d\lambda}{d\zeta}(r\delta r - z\delta z) = \int\delta.\frac{dR}{d\lambda}d\zeta + 2\frac{\pi}{n}\int xdz,$$

$$\frac{d^2\delta z}{d\zeta^2} + \left(\frac{\mu}{n^2r^3} + m^2\right)\delta z = -2\frac{\pi}{n}\frac{dy}{d\zeta}.$$

In these equations terms multiplied by the square and higher powers of the inclination of the moon's orbit are neglected; and, since δr and $\delta\lambda$ are multiplied by the first power of this quantity, this involves the neglect of terms such as $z\delta r$ and $z\delta\lambda$. For the same reason all higher powers of z than the second have been omitted in R.

These equations suffice to determine the inequalities we seek; but for a term of long period in $\delta\lambda$ it will be more commodious to employ another equation. We have

$$r\frac{d^2r}{d\zeta^2} - (r^2 - z^2)\frac{d\lambda^2}{d\zeta^2} - \left(\frac{dz}{d\zeta} - \frac{z}{r}\frac{dr}{d\zeta}\right)^2 + \frac{\mu}{n^2r} = r\frac{dR}{dr} + 2\frac{\pi}{n}\frac{ydz - zdy}{d\zeta},$$

or

$$\frac{r^2-z^2}{r^2}\frac{d\lambda^2}{d\zeta^2}-\frac{1}{r}\frac{d^2r}{d\zeta^2}+\left(\frac{1}{r}\frac{dz}{d\zeta}-\frac{z}{r^2}\frac{dr}{d\zeta}\right)^2-\frac{\mu}{n^2r^3}=-2\frac{R}{r^2}-2\frac{\pi}{n}\frac{ydz-zdy}{r^2d\zeta}.$$

Taking the variation with respect to π, and then multiplying by r^2,

$$\left\{\begin{array}{l} 2r^2\frac{d\lambda}{d\zeta}\frac{d.\delta\lambda}{d\zeta}-2\frac{d\lambda^2}{d\zeta^2}z\delta z-\frac{rd^2\delta r-d^2r.\delta r}{d\zeta^2} \\ +2\left(\frac{dz}{d\zeta}-\frac{z}{r}\frac{dr}{d\zeta}\right)\left(\frac{d.\delta z}{d\zeta}-\frac{dr\delta z}{rd\zeta}\right)+\frac{3\mu r\delta r}{n^2r^3}\end{array}\right\}=\left\{\begin{array}{l}-2\frac{dR}{d\lambda}\delta\lambda-2\frac{dR}{dz}\delta z \\ -2\frac{\pi}{n}\frac{ydz-zdy}{d\zeta}\end{array}\right\}$$

But

$$3\frac{d^2r\delta r}{d\zeta^2}+3\frac{\mu r\delta r}{n^2r^3}=12\delta R+6m\int\delta.\frac{dR}{d\lambda}d\zeta+6\frac{\pi}{n}\frac{ydz-zdy}{d\zeta};$$

subtracting this

$$r^2\frac{d\lambda}{d\zeta}\frac{d.\delta\lambda}{d\zeta}=\frac{d[2d(r\delta r)-dr.\delta r]}{d\zeta^2}+\frac{d\lambda^2}{d\zeta^2}z\delta z-\frac{dz}{d\zeta}\frac{d.\delta z}{d\zeta}+\frac{d(z\delta z)}{d\zeta}\frac{dr}{rd\zeta}$$
$$-\left(\frac{dr}{rd\zeta}\right)^2z\delta z-7\delta R+\frac{dR}{dr}\delta r-3m\int\delta.\frac{dR}{d\lambda}d\zeta-4\frac{\pi}{n}\frac{ydz-zdy}{d\zeta}.$$

In determining δz we shall stop at terms of the order $m\frac{\pi}{n}$, and shall neglect all terms multiplied by powers of the lunar eccentricity e higher than the first. In $\delta\lambda$ we shall neglect e altogether; and, since the inequalities in the lunar parallax resulting from δr are insensible, δr will be determined only so far as it is necessary to the determination of $\delta\lambda$. Let ξ denote the moon's mean anomaly, and η its mean argument of latitude, or $\eta=\varepsilon+nt-\Omega$. In applying the last equation to determining the co-efficient of sin $(\zeta-\eta)$ in $\delta\lambda$ to terms of the order of $\gamma\frac{\pi}{n}$, (where γ denotes the same function of the inclination as it does in Pontécoulant's *Théorie Analytique*), it will be necessary to compute each member to terms of the order of $m^2\gamma\frac{\pi}{n}$. But $r\delta r$ is of the order of $\gamma\frac{\pi}{n}$, consequently $\frac{d^2r\delta r}{d\zeta^2}$, in the term which has $\zeta-\eta$ for its argument, is of the order of $m^4\gamma\frac{\pi}{n}$, and thus may be neglected; moreover $\frac{dr}{d\zeta}$ is of the order of m^2, and hence $\frac{d(dr.\delta r)}{d\zeta^2}$ is of the order of $m^4\gamma\frac{\pi}{n}$ in the term having the same argument; this may then be also omitted.

With these simplifications the last equation becomes

$$r^2\frac{d\lambda}{d\zeta}\frac{d.\delta\lambda}{d\zeta}=\frac{d\lambda^2}{d\zeta^2}z\delta z-\frac{dz}{d\zeta}\frac{d.\delta z}{d\zeta}+\frac{dr}{rd\zeta}\frac{d(z\delta z)}{d\zeta}-\left(\frac{dr}{rd\zeta}\right)^2z\delta z$$
$$+\frac{21}{2}m^2(1+\cos 2\varphi)z\delta z-4\frac{\pi}{n}\frac{ydz-zdy}{d\zeta}$$
$$-3m^2(1+3\cos 2\varphi)r\delta r-7\frac{dR}{d\lambda}\delta\lambda-3m\int\delta.\frac{dR}{d\lambda}d\zeta.$$

If, for brevity, we write

$$
\begin{aligned}
A &= \frac{\mu}{n^2 r^3} + m^2, \\
B &= \frac{\mu}{n^2 r^3} - 2m^2 - 6m^2 \cos 2\varphi, \\
C &= 6m^2 r^2 \sin 2\varphi, \\
D &= 3m^2 r^2 \cos 2\varphi, \\
E &= 3m^2 \sin 2\varphi, \\
U &= -2\frac{\pi}{n}\frac{dy}{d\zeta}, \\
U' &= 2\frac{\pi}{n}\frac{ydz - zdy}{d\zeta} - 6m^2 (1 + \cos 2\varphi)\, z\delta z + 6m^3 \int \sin 2\varphi .\, z\delta z\, d\zeta, \\
U'' &= 2\frac{\pi}{n}\int x dz + 2\frac{d\lambda}{d\zeta} z\delta z + 3m^2 \int \sin 2\varphi . z\delta z d\zeta,
\end{aligned}
$$

the term $2m\int Er\delta r d\zeta$ in the equation for $r\delta r$ being omitted as not giving any terms which we wish to preserve, and it being sufficient to put $B = 1$, and $\frac{d\lambda}{d\zeta} = 1$, where the latter multiplies $r\delta r$ in the equation for $\delta\lambda$, the three equations become

$$
\begin{aligned}
&\frac{d^2 \delta z}{d\zeta^2} + A\delta z &&= U, \\
&\frac{d^2 r\delta r}{d\zeta^2} + r\delta r + C\delta\lambda + 2m\int D\delta\lambda d\zeta &&= U', \\
&r^2 \frac{d.\delta\lambda}{d\zeta} + 2\, r\delta r + \int [D\delta\lambda + Er\delta r]\, d\zeta &&= U''.
\end{aligned}
$$

To the degree of approximation we desire

$$
\begin{aligned}
\frac{1}{r} &= 1 + \frac{1}{6}m^2 + (m^2 + \frac{19}{6}m^3) \cos 2\tau, \\
\lambda &= \varepsilon + nt + \left(\frac{11}{8}m^2 + \frac{59}{12}m^3\right) \sin 2\tau.
\end{aligned}
$$

Also (*Pontécoulant, Theorie Analytique*, Tom. IV., pp. 216, 226)

$$
\begin{aligned}
A = 1 + \frac{3}{2}m^2 - \frac{9}{32}m^4 + \frac{55}{16}m^5 + (3m^2 + \frac{19}{2}m^3 + \frac{137}{6}m^4) \cos 2\tau \\
+ (3 + \frac{3}{2}m^2)\, e \cos \xi + \left(\frac{45}{8}m + \frac{657}{32}m^3\right) e \cos (2\tau - \xi) + \frac{147}{16}m^2 e \cos (2\tau + \xi).
\end{aligned}
$$

From $y = r\sin(\lambda - \Pi)$ we derive

$$y = -\left(1 - \frac{m^2}{6}\right)\cos\zeta + \frac{19}{16}m^2\cos(\zeta - 2\tau) - \frac{1}{2}e\cos(\zeta + \xi),$$

$$U = -\left(2 - \frac{m^2}{3}\right)\frac{\pi}{n}\sin\zeta - \frac{19}{8}m^2\frac{\pi}{n}\sin(\zeta - 2\tau) - 2e\frac{\pi}{n}\sin(\zeta + \xi).$$

Let

$$\begin{aligned}\delta z = \frac{\pi}{n}\Big\{ & A_1\sin\zeta + A_2\sin(\zeta - 2\tau) + A_3\sin(\zeta + 2\tau) + A_4\sin(\zeta - 4\tau) \\ & + A_5 e\sin(\zeta - \xi) + A_6 e\sin(\zeta + \xi) + A_7 e\sin(\zeta - 2\tau + \xi) + A_8 e\cos(\zeta + 2\tau - \xi) \\ & + A_9 e\sin(\zeta - 2\tau - \xi) + A_{10} e\sin(\zeta + 2\tau + \xi) + A_{11} e\sin(\zeta - 4\tau + \xi)\Big\}.\end{aligned}$$

On substituting this expression in the first of the three differential equations, the following equations result for determining A_1, A_2, etc.,

$$\begin{aligned}
&\left(\frac{3}{2}m^2 - \frac{9}{32}m^4 + \frac{55}{16}m^5\right)A_1 + \left(\frac{3}{2}m^2 + \frac{19}{4}m^3 + \frac{137}{12}m^4\right)(A_2 + A_3) = -2 + \frac{1}{3}m^2, \\
&\left(4m - \frac{5}{2}m^2\right)A_2 + \left(\frac{3}{2}m^2 + \frac{19}{4}m^3 + \frac{137}{12}m^4\right)A_1 = -\frac{19}{8}m^2, \\
&-(8 - 12m)A_3 + \left(\frac{3}{2}m^2 + \frac{19}{4}m^3\right)A_1 = 0, \\
&-8A_4 + \frac{3}{2}m^2 A_2 = 0, \\
&\left(1 + \frac{3}{2}m^2\right)A_5 + \left(\frac{3}{2} + \frac{3}{4}m\right)A_1 + \frac{45}{16}mA_2 = 0, \\
&-\left(3 - \frac{9}{2}m^2\right)A_6 + \left(\frac{3}{2} + \frac{3}{4}m\right)A_1 = -2, \\
&A_7 + \frac{3}{2}m^2 A_6 + \frac{3}{2}A_2 + \left(\frac{45}{16}m + \frac{657}{64}m^2\right)A_1 = 0, \\
&-(3 - 8m)A_8 + \frac{3}{2}m^2 A_5 + \frac{3}{2}A_3 + \left(\frac{45}{16}m + \frac{657}{64}m^2\right)A_1 = 0, \\
&-(3 - 8m)A_9 + \frac{3}{2}m^2 A_5 + \frac{3}{2}A_2 + \frac{147}{32}m^2 A_1 = 0, \\
&-15A_{10} + \frac{3}{2}m^2 A_6 + \frac{3}{2}A_3 + \frac{147}{32}m^2 A_1 = 0, \\
&-3A_{11} + \frac{45}{16}mA_2 = 0.
\end{aligned}$$

By the solution of these, this expression of δz is obtained.

$$\begin{aligned}\delta z = & -\left(\frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{31}{9} + \frac{3395}{288}m\right)\frac{\pi}{n}\sin\zeta \\ & + \left(\frac{1}{2}m^{-1} + \frac{25}{12} + \frac{1843}{288}m\right)\frac{\pi}{n}\sin(\zeta - 2\tau)\end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{1}{4} + \frac{121}{96} m\right) \frac{\pi}{n} \sin(\zeta + 2\tau) \\
&+ \frac{3}{32} m \frac{\pi}{n} \sin(\zeta - 4\tau) \\
&+ \left(2 m^{-2} + \frac{3}{4} m^{-1} + \frac{169}{96}\right) e \frac{\pi}{n} \sin(\zeta - \xi) \\
&- \left(\frac{2}{3} m^{-2} + \frac{1}{4} m^{-1} + \frac{43}{18}\right) e \frac{\pi}{n} \sin(\zeta + \xi) \\
&+ \left(3 m^{-1} + \frac{415}{32}\right) e \frac{\pi}{n} \sin(\zeta - 2\tau + \xi) \\
&- \left(\frac{5}{4} m^{-1} + \frac{719}{96}\right) e \frac{\pi}{n} \sin(\zeta + 2\tau - \xi) \\
&+ \left(\frac{1}{4} m^{-1} - \frac{9}{24}\right) e \frac{\pi}{n} \sin(\zeta - 2\tau - \xi) \\
&- \frac{1}{2} e \frac{\pi}{n} \sin(\zeta + 2\tau + \xi) \\
&+ \frac{15}{32} e \frac{\pi}{n} \sin(\zeta - 4\tau + \xi).
\end{aligned}
$$

The value of z (*Théorie Analytique*, Tom. IV., pp. 237, 244) is

$$
\begin{aligned}
z = \gamma \Big\{ & \left(1 - \frac{m^2}{6} + \frac{57}{128} m^3\right) \sin \eta + \left(\frac{3}{8} m + \frac{41}{32} m^2 + \frac{5293}{1536} m^3\right) \sin(2\tau - \eta) \\
& + \left(\frac{3}{16} m^2 + \frac{7}{8} m^3\right) \sin(2\tau + \eta) \Big\},
\end{aligned}
$$

whence, by multiplication, is obtained

$$
\begin{aligned}
z\delta z = &- \left(\frac{2}{3} m^{-2} + \frac{1}{4} m^{-1} + \frac{491}{288}\right) \gamma \frac{\pi}{n} \cos(\zeta - \eta) \\
&+ \left(\frac{1}{4} m^{-1} + \frac{11}{12} + \frac{91}{36} m\right) \gamma \frac{\pi}{n} \cos(\zeta - \eta - 2\tau) \\
&+ \left(\frac{1}{4} m^{-1} + \frac{79}{76} + \frac{6067}{2304} m\right) \gamma \frac{\pi}{n} \cos(\zeta - \eta + 2\tau) \\
&+ \left(\frac{2}{3} m^{-2} + \frac{1}{4} m^{-1} + \frac{29}{18}\right) \gamma \frac{\pi}{n} \cos(\zeta + \eta) \\
&- \left(\frac{1}{2} m^{-1} + \frac{191}{96} + \frac{14795}{2304}\right) \gamma \frac{\pi}{n} \cos(\zeta + \eta - 2\tau) \\
&+ \frac{1}{4} \gamma \frac{\pi}{n} \cos(\zeta + \eta + 2\tau) \\
&+ \frac{3}{32} \gamma \frac{\pi}{n} \cos(\zeta + \eta - 4\tau).
\end{aligned}
$$

[TO BE CONTINUED.]